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1998 J. Phys. A: Math. Gen. 31 9769

(http://iopscience.iop.org/0305-4470/31/48/011)

J. Phys. A: Math. Gen. **31** (1998) 9769–9772. Printed in the UK PII: S0305-4470(98)94515-2

Generation of self-fractional Hankel functions

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Received 22 May 1998, in final form 9 September 1998

Abstract. A definition of the self-fractional Hankel functions and a formula for generating them are presented. Some properties of these functions are investigated. It is shown that a self-fractional Hankel function for an angle $\alpha = \pi N/M$, where N and M are indivisible integers ($N < M$), is also a self-fractional Hankel function for angles $\pi j/M$ ($j = 1, 2, 3 \ldots$).

1. Introduction

The Hankel transform (HT), like the Fourier transform (FT) and the Laplace transform, is a widely applicable mathematical tool in physics and other fields [1]. For example, the zero-order HT describes the diffraction effect of an axially symmetric light beam in free space and the high-order HTs are usually used in the analysis of a laser cavity with circular mirrors. Although the fractional Fourier transform (FRFT) was proposed earlier in [2–4], the fractional Hankel transform (FRHT) was only introduced recently [5, 6]. The FRHT is effectively used in the design of lenses, the analysis of a laser cavity, the study of wave propagation in a quadratic refractive index (GRIN) medium when the system is axially symmetric.

In this paper, we define the self-fractional Hankel functions (SFHFs) as eigenfunctions of the FRHT for some angles, and propose a formula for constructing the SFHF. We also investigate some properties of SFHFs and indicate that a SFHF with any given angle α is also a SFHF for angles $j\beta$, where β is a certain angle depending on α and $j = 1, 2, 3 \ldots$.

The FRHT of a function $f(r)$ for an angle α is defined as follows [5, 6]:

$$H_v^\alpha\{f(r)\} = \int_0^\infty f(r) K_v^\alpha(r, \rho) r \, \mathrm{d}r \tag{1}$$

where the kernel

$$K_v^\alpha(r, \rho) = \frac{\exp[\mathrm{i}(1 + v)((\pi/2) - \alpha)]}{\sin\alpha} \exp\left[-\mathrm{i}\frac{1}{2}(r^2 + \rho^2)\cot\alpha\right] J_v\left(\frac{r\rho}{\sin\alpha}\right) \tag{2}$$

where J_v is the vth-order Bessel function and v is an integer. This transform describes the behaviour of wave propagation through an axially symmetric GRIN medium in the paraxial approximation. α changes from 0 to π for $\alpha < 0$ and for $\alpha > \pi$, and the periodicity property

$$H_v^{\alpha+\pi}\{f(r)\} = H_v^\alpha\{f(r)\} \tag{3}$$

can be used to express the FRHT in the region of $0 < \alpha < \pi$. If α is a multiple of π, the kernel $K_v^\alpha(r, \rho)$ corresponds to $\delta(r - \rho)$, and the FRHT reduces to the identity operator.

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For $\alpha = \pi/2$, the FRHT becomes the conventional HT. The FRHT possesses commutative additivity and linearity properties as follows [6]:

$$H_v^\alpha\{H_v^\beta\{f(r)\}\} = H_v^\beta\{H_v^\alpha\{f(r)\}\} = H_v^{\alpha+\beta}\{f(r)\}. \tag{4}$$

$$H_v^\alpha\{c_1 f(r) + c_2 g(r)\} = c_1 H_v^\alpha\{f(r)\} + c_2 H_v^\alpha\{g(r)\} \tag{5}$$

where c_1 and c_2 are complex constants.

Similarly to the definition of the self-fractional Fourier function [7, 8], we define such a function as a SFHF for angle α if it satisfies the following equation:

$$H_v^\alpha\{f_v^\alpha(r)\}(\rho) = A f_v^\alpha(\rho) \tag{6}$$

where A is a complex constant factor. Equation (6) means that $f_v^\alpha(r)$ is an eigenfunction of the corresponding FRHT operator H_v^α with an eigenvalue A. From equations (1), (2) and (4), it is evident that any function is certainly a SFHF for $\alpha = n\pi$; however, a SFHF should be a self-Hankel function (SHF) when $\alpha = n\pi/2$, where n is an integer. Using the Parserval relation of the FRHT [6]

$$\int_0^\infty |f_v^\alpha(r)|^2 r\,\mathrm{d}r = \int_0^\infty |H_v^\alpha\{f_v^\alpha(r)\}(\rho)|^2 \rho\,\mathrm{d}\rho = |A|^2 \int_0^\infty |f_v^\alpha(\rho)|^2 \rho\,\mathrm{d}\rho \tag{7}$$

it follows that $|A| = 1$, i.e. A should have the form $A = \exp(\pm \mathrm{i}2\pi\phi)$, where ϕ is a real constant. This means that all SFHFs can image themselves after performing the FRHT with the corresponding angle α. When $A = 1$, it is the case of exact self-reproduction.

Equation (3) shows that the FRHT possesses periodicity with a period of π; therefore, for angle $\alpha \neq n\pi$, it can be expressed in the form $\alpha = \pi N/M$, where N and M are irreducible integers and $N < M$. Taking into account the periodicity property (3), any function retrieves itself after applying a cascade of K times a FRHT for angle α,

$$H_v^{K\alpha}\{f(r)\} = f(r) \tag{8}$$

where K and L are the smallest integers satisfying the following condition:

$$KN/M = L. \tag{9}$$

As N and M are irreducible and $N < M$, therefore the smallest integer K satisfying equation (8) should be M. Combining equations (4), (6) and (8), it follows that $A^M = \exp(\mathrm{i}2\pi\phi M) = 1$, consequently ϕ takes a series of values j/M $(j = 1, 2, 3, \ldots)$.

The above analysis indicates a way for generating a SFHF with angle $\alpha = \pi N/M$. For instance, for any transformable function $g(r)$, one can use its FRHT spectra with a sequence of angles and construct a linear superposition of them with appropriate coefficients,

$$f_v^\alpha(r) = \sum_{k=1}^{K} \exp(\mp \mathrm{i}2\pi k\phi) H_v^{(k-1)\alpha}\{g(\rho)\}. \tag{10}$$

Using equations (3) and (4), and $K\alpha = K\pi N/M = L\pi$, once can easily prove that

$$\begin{aligned} H_v^\alpha\{f_v^\alpha(r)\}(\rho) &= \sum_{k=1}^{K} \exp(\mp \mathrm{i}2\pi k\phi) H_v^\alpha\{H_v^{(k-1)\alpha}\{g(\rho)\}\}(\rho) \\ &= \sum_{k=1}^{K} \exp[\mp \mathrm{i}2\pi(k-1)\phi] H_v^{(k-1)\alpha}\{g(r)\} \\ &= \exp(\pm \mathrm{i}2\pi\phi) f_v^\alpha(\rho). \end{aligned} \tag{11}$$

Thus, the corresponding eigenvalue for this special constructing SFHF is $\exp(\pm \mathrm{i}2\pi\phi)$ under the FRHT operator with angle $\alpha = \pi N/M$. When ϕ is an integer and the eigenvalue $A = 1$, the SFHF becomes the exact self-reproduction after performing the FRHT with $\alpha = \pi N/M$. When $\alpha = \pi/2$, the FRHT reduces to the classical HT and the corresponding SHF can be expressed in the form

$$f_v^{\pi/2}(r) = g(r) + H_v^{\pi/2}\{g(r)\}.$$

If the function $g(r)$ is selected as $g(r) = e^{-r}$, for $v = 0$, the SHF $f_0^{\pi/2}(r)$ has the form

$$f_0^{\pi/2}(r) = e^{-r} + \frac{1}{\sqrt{(r^2+1)^3}}$$

and for $v = 1$, the SHF $f_1^{\pi/2}(r)$ takes the form

$$f_1^{\pi/2}(r) = e^{-r} + \frac{r}{\sqrt{(r^2+1)^3}}.$$

The SFHFs have some specific features, which may be useful for the analysis of self-imaging in the FRHT optical system. We now investigate some of their fundamental properties.

The linear superposition of SFHFs for the same angle with equal eigenvalue A is also a SFHF for this angle with eigenvalues A. Suppose that both $f_v^{\alpha}(r)$ and $h_v^{\alpha}(r)$ are SFHFs for angle α with eigenvalue A; by using the linearity of FRHT (5) and definition (6), we have

$$H_v^{\alpha}\{c_1 f_v^{\alpha}(r) + c_2 h_v^{\alpha}(r)\} = A[c_1 f_v^{\alpha}(r) + c_2 h_v^{\alpha}(r)]. \tag{12}$$

The FRHT of a SFHF for angle α with eigenvalue A is also a SFHF for angle α with the same eigenvalue. From equation (4) and definition (6)

$$H_v^{\alpha}\{H_v^{\beta}\{f_v^{\alpha}(r)\}\} = H_v^{\beta}\{H_v^{\alpha}\{f_v^{\alpha}(r)\}\} = AH_v^{\beta}\{f_v^{\alpha}(r)\}. \tag{13}$$

It is found that a SFHF for any angle $\alpha = \pi N/M$ is also a SFHF for the angle $\beta = \pi/M$ and *vice versa*. From equation (4), it is seen that if a function is a SFHF for $\alpha = \pi N/M$, then it is also a FRHT for $j\alpha = \pi jN/M$ ($j = 1, 2, 3\ldots$):

$$\begin{aligned} H_v^{j\alpha}\{f_v^{\alpha}(r)\} &= H_v^{(j-1)\alpha}\{H_v^{\alpha}\{f_v^{\alpha}(\rho)\}\} \\ &= AH_v^{(j-1)\alpha}\{f_v^{\alpha}(r)\} = \cdots = A^j f_v^{\alpha}(\rho). \end{aligned} \tag{14}$$

From equation (14), it is evident that $f_v^{\alpha}(r)$ is also an eigenfunction of the FRHT for angle $j\alpha$, and the corresponding eigenvalue is A^j. Because the FRHT is a periodic operator, for some j, one has $j\alpha = \pi jN/M = n\pi + \pi/M$; thus the corresponding FRHT reduces to the FRHT with angle $\beta = \pi/M$. Consequently, a SFHF with angle $\alpha = \pi N/M$ also is certainly a SFHF with angle $\beta = \pi/M$. If $N = 1$ is selected in equation (14), it is easy to find that if a function is a SFHF with $\beta = \pi/M$, this function is also a FRHT with angle $\alpha = \pi N/M$. Furthermore, this function is also a SFHF for angles $\pi j/M$ ($j = 1, 2, 3, \ldots$). We have indicated some fundamental properties of SFHFs, further research should be stimulated by applications of the FRHT system.

In general, for any linear integral transform R with periodicity T and additivity properties, as soon as it satisfies the Parseval relation (7), there exists a general method for generating the eigenfunctions of this transform from any transformable function $g(x)$. The eigenfunctions are defined by an equation similar to equation (6). For any order $\alpha \neq nT$, it can always be expressed in the form $\alpha = NT/M$, where N and M are irreducible and $N < M$. Using the Parserval relation, the eigenvalue A takes $|A| = 1$. Combining the periodicity and additivity properties, one can derive $A^M = \exp(\mathrm{i}2\pi\phi M) = 1$ and thus the eigenfunctions of this transform with order α are

$$F^{\alpha}(x) = \sum_{k=1}^{K} \exp(\mp \mathrm{i}2\pi\phi) R^{(k-1)\alpha}\{g(x)\} \tag{15}$$

where K and L are the smallest integers satisfying equation (9). ϕ can take a series of values, j/M ($j = 1, 2, 3, \ldots$). The self-fractional Fourier functions and SFHFs can be considered as particular cases.

In conclusion, we have proposed a definition of the SFHFs and a procedure to generate such SFHFs for any angle α with eigenvalue $A = \exp(\pm i2\pi j/M)$. Some properties of these functions have been investigated. It has been proven that a SFHF for angle $\alpha = \pi N/M$ is also a SFHF for angles $\beta = \pi j/M$ ($j = 1, 2, 3, \ldots$). We have also generalized our results to the case of eigenfunctions of any linear integral transform in which the periodicity and additivity properties, as well as the Parseval relation, hold.

Acknowledgment

This work was supported by the National Natural Science Foundation of China.

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